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Non-linear waves in a two-component hot plasma

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Abstract. We study wave propagation in a hot collisionless two-component plasma by working in a Lorentz frame of reference in which the spatial dependence is eliminated and where there is no ambient magnetic field. We obtain non-linear dispersion relations which govern transverse propagation for the cases: (i) when the temperature correction is small but the wave amplitude is large; (ii) when the wave amplitude is small and the temperature is arbitrary.

1. Introduction

Following Clemmow (1974, 1975), we study non-linear wave propagation in a hot, collisionless plasma consisting of electrons and ions. We assume that the plasma is unbounded and that there is no ambient magnetic field. The model used is the Boltzmann–Vlasov equations (BV equations) in a Lorentz frame of reference S in which the space-dependence is eliminated (Winkles and Eldridge 1972).

We investigate tranverse waves for the two following cases, using a perturbation technique. (i) When the plasma is not extremely hot so that the temperature effect can be treated as a small correction to the cold plasma case and the amplitude of the wave is large. (ii) When the wave amplitude is small and there is no restriction on the temperature. In both cases, the dispersion relations are obtained and the results are presented in such a way that the electron and ion effects stand out separately.

The plan of the paper is as follows: § 2 presents a general formulation of the problem, while § 3 specializes to transverse propagation and develops the master equations (15) and (16). In § 3.1 we record the results for the cold plasma. The main results of this paper are given in § 3.2 and § 3.3 which describe the dispersion relations for strong waves (i.e. large amplitude) with first-order temperature effect, and for weak waves (i.e. small amplitude) with first-order non-linear correction, keeping temperature arbitrary.

2. General formulation

We consider S′ as the laboratory frame in which the velocity of the wave is $(0, 0, c/n)$ and S is the frame in which there is no space dependence and which is moving with velocity $(0, 0, nc)$ relative to S′ (n being the refractive index of the medium). All our calculations will be in frame S which can then be transformed to frame S′ with the help of a Lorentz transformation.

Due to the absence of the spatial dependence of the fields in frame S, Maxwell's equations imply that the magnetic field $\boldsymbol{B}$ is constant and that the number densities of electrons and ions are equal, say N. Further, the curl of $\boldsymbol{B}$ equation is reduced to

$$-\epsilon_0\dot{\boldsymbol{E}} = \sum_{\alpha=\mathrm{e,i}} \boldsymbol{J}_\alpha. \tag{1}$$

We consider the special case $\boldsymbol{B} = 0$. Then the relativistic BV equations for electrons and ions will be

$$\frac{\partial f_\alpha}{\partial t} + \frac{q_\alpha}{m_\alpha c}\boldsymbol{E}\cdot\frac{\partial f_\alpha}{\partial \boldsymbol{u}_\alpha} = 0, \tag{2}$$

where $\boldsymbol{u}_\mathrm{e}$ and $\boldsymbol{u}_\mathrm{i}$ are the reduced velocities of electrons and ions respectively, defined in terms of the ordinary velocities $\boldsymbol{v}_\mathrm{e}$ and $\boldsymbol{v}_\mathrm{i}$ by

$$\boldsymbol{u}_\alpha = \frac{\gamma_\alpha \boldsymbol{v}_\alpha}{c}, \qquad \gamma_\alpha = (1+\boldsymbol{u}_\alpha^2)^{1/2}.$$

Also $Nf_\alpha(\boldsymbol{u}_\alpha, t)$ is the distribution function. Now, using $\boldsymbol{E} = -\dot{\boldsymbol{A}}$ and defining $\lambda_\alpha = -q_\alpha/m_\alpha c$ where $q_\mathrm{i} = +e$ and $q_\mathrm{e} = -e$, the BV equation may be expressed as

$$\frac{\partial f_\alpha}{\partial t} + \lambda_\alpha\dot{\boldsymbol{A}}\cdot\frac{\partial f_\alpha}{\partial \boldsymbol{u}_\alpha} = 0. \tag{3}$$

These equations have a general solution

$$f_\alpha(\boldsymbol{u}_\alpha, t) = F_\alpha(\boldsymbol{u}_\alpha - \lambda_\alpha\boldsymbol{A}) \tag{4}$$

where F_α is an arbitrary function of its argument; and $\boldsymbol{u}_\alpha - \lambda_\alpha\boldsymbol{A}$ is $1/m_\alpha c$ times the generalized momentum. Now

$$\boldsymbol{J}_\alpha = Nq_\alpha c\int \frac{\boldsymbol{u}_\alpha}{\gamma_\alpha} f_\alpha(\boldsymbol{u}_\alpha, t)\,\mathrm{d}^3\boldsymbol{u}_\alpha = \frac{Nq_\alpha c}{\lambda_\alpha}\frac{\partial V_\alpha}{\partial \boldsymbol{A}}$$

where

$$V_\alpha = \int [1+(\boldsymbol{u}_\alpha + \lambda_\alpha\boldsymbol{A})^2]^{1/2} F_\alpha(\boldsymbol{u}_\alpha)\,\mathrm{d}^3\boldsymbol{u}_\alpha. \tag{5}$$

Equation (1) may therefore be rewritten as

$$\ddot{\boldsymbol{A}} + \sum_\alpha \frac{\omega_\alpha^2}{\lambda_\alpha^2}\frac{\partial V_\alpha}{\partial \boldsymbol{A}} = 0 \tag{6}$$

where $\omega_\alpha^2 = Nq_\alpha^2/\epsilon_0 m_\alpha$.

The form of equation (6) is identical with that in Newtonian dynamics of a particle of unit mass with position vector $\boldsymbol{A} = (A_x, A_y, A_z)$ under the conservative force field of potential

$$V = \sum_\alpha \frac{\omega_\alpha^2}{\lambda_\alpha^2} V_\alpha. \tag{7}$$

We may now rewrite equation (6) as

$$\ddot{\boldsymbol{A}} + \frac{\partial V}{\partial \boldsymbol{A}} = 0. \tag{8}$$

Introducing cylindrical polar coordinates

$$\boldsymbol{A} = (\mathscr{A} \cos \Phi, \mathscr{A} \sin \Phi, A_z) \tag{9}$$

and assuming that the function $F(\boldsymbol{u})$ is isotropic so that the potential V is a function of $\mathscr{A}$ and A_z only, the above expression becomes

$$\ddot{\mathscr{A}} - \frac{h^2}{\mathscr{A}^3} + \frac{\partial V}{\partial \mathscr{A}} = 0 \tag{10}$$

$$\ddot{A}_z + \frac{\partial V}{\partial A_z} = 0 \tag{11}$$

$$\mathscr{A}^2 \dot{\Phi} = h \tag{12}$$

where h is a constant.

3. Transverse waves

For pure transverse waves, $E_z = 0$ so that A_z is constant. In this case equation (11) in turn implies that $\mathscr{A}$ is also a constant. With constant $\mathscr{A}$, equation (12) under appropriate initial conditions has a solution

$$\Phi = \omega t; \qquad \omega = h/\mathscr{A}^2 \tag{13}$$

and equation (10) reduces to the form

$$\frac{\partial V}{\partial \mathscr{A}} = \frac{h^2}{\mathscr{A}^3} \qquad \text{or} \qquad \frac{1}{\mathscr{A}} \frac{\partial V}{\partial \mathscr{A}} = \omega^2. \tag{14}$$

There is thus in the S frame a monochromatic circularly polarized field of vector potential

$$\boldsymbol{A} = [\mathscr{A} \cos(\omega t), \mathscr{A} \sin(\omega t), A_z]$$

and

$$\boldsymbol{E} = \mathscr{A}\omega [\sin(\omega t), -\cos(\omega t), 0]$$

where $\mathscr{A}$, A_z and ω satisfy the equations

$$\sum_\alpha \frac{\omega_\alpha^2}{\lambda_\alpha^2} \frac{\partial V_\alpha}{\partial A_z} = 0 \tag{15}$$

$$\sum_\alpha \frac{\omega_\alpha^2}{\lambda_\alpha^2} \frac{1}{\mathscr{A}} \frac{\partial V_\alpha}{\partial \mathscr{A}} = \omega^2. \tag{16}$$

Transforming the results to the laboratory frame S′ again yields a purely transverse circularly polarized wave with velocity $(0, 0, c/n)$ and angular frequency ω'. The fields in the laboratory frame S′ will be

$$\boldsymbol{E}' = E_0'\{\sin[\omega'(t' - nz'/c)], -\cos[\omega'(t' - nz'/c)], 0\}$$

and

$$\boldsymbol{B}' = \frac{n}{c} \hat{\boldsymbol{z}} \times \boldsymbol{E}'$$

where the electric field amplitudes in S and S′ are related by

$$E_0/\omega = E_0'/\omega' = \mathscr{A}. \tag{17}$$

The dispersion relation is obtained by determining ω in terms of $\mathscr{A}$ from (15) and (16) and then substituting it in

$$\omega = (1-n^2)^{1/2}\omega'. \tag{18}$$

3.1. Dispersion relation in cold plasma

The cold plasma results can be obtained by taking anisotropic streaming distributions, i.e.

$$F_\alpha(\boldsymbol{u}_\alpha) = \delta(\xi_\alpha)\delta(\eta_\alpha)\delta(\zeta_\alpha - u_{\alpha 0}) \tag{19}$$

where

$$\boldsymbol{u}_\alpha = (\xi_\alpha, \eta_\alpha, \zeta_\alpha) = (\rho_\alpha \cos\phi_\alpha, \rho_\alpha \sin\phi_\alpha, \zeta_\alpha) \tag{20}$$

and $u_{\alpha 0}$ is the reduced streaming velocity given by

$$u_{\alpha 0} = \frac{v_{\alpha 0}}{c}\left(1 - \frac{v_{\alpha 0}^2}{c^2}\right)^{-1/2} = \frac{\gamma_{\alpha 0} v_{\alpha 0}}{c}. \tag{21}$$

The velocities u_{e0} and u_{i0} are related through the momentum conservation equation as

$$u_{i0} + \mu u_{e0} = u_0(\text{constant}), \qquad \mu = m_e/m_i. \tag{22}$$

The function V_α now takes a simpler form

$$V_\alpha = [1 + \lambda_\alpha^2 \mathscr{A}^2 + (u_{\alpha 0} + \lambda_\alpha A_z)^2]^{1/2} \equiv \Delta_\alpha \tag{23}$$

say, so that

$$\frac{1}{\lambda_\alpha}\frac{\partial V_\alpha}{\partial A_z} = \frac{\mathscr{L}_\alpha}{\Delta_\alpha}; \qquad \mathscr{L}_\alpha = u_{\alpha 0} + \lambda_\alpha A_z \tag{24}$$

and

$$\frac{1}{\lambda_\alpha^2 \mathscr{A}}\frac{\partial V_\alpha}{\partial \mathscr{A}} = \frac{1}{\Delta_\alpha}. \tag{25}$$

Now observing $\omega_i^2 = \mu\omega_e^2$ and $\lambda_i = -\mu\lambda_e$, and using equations (24) and (25), equations (15) and (16) become

$$\frac{\mathscr{L}_e}{\Delta_e} - \frac{\mathscr{L}_i}{\Delta_i} = 0 \tag{26}$$

and

$$\frac{1}{\Delta_e} + \frac{\mu}{\Delta_i} = \frac{\omega^2}{\omega_e^2}. \tag{27}$$

Also

$$\mathscr{L}_\alpha = (1 + \lambda_\alpha^2 \mathscr{A}^2)^{1/2}\Omega \tag{28}$$

where

$$\Omega = \frac{u_0}{(1+\lambda_i^2\mathscr{A}^2)^{1/2} + \mu(1+\lambda_e^2\mathscr{A}^2)^{1/2}}. \tag{29}$$

Therefore

$$\Delta_\alpha = (1+\lambda_\alpha^2\mathscr{A}^2 + \mathscr{L}_\alpha^2)^{1/2} = (1+\lambda_\alpha^2\mathscr{A}^2)^{1/2}(1+\Omega^2)^{1/2}. \tag{30}$$

Thus the dispersion relation (27) becomes

$$\frac{\omega^2}{\omega_e^2} = \frac{1}{(1+\Omega^2)^{1/2}}\left(\frac{1}{(1+\lambda_e^2\mathscr{A}^2)^{1/2}} + \frac{\mu}{(1+\lambda_i^2\mathscr{A}^2)^{1/2}}\right). \tag{31}$$

Note that the ionic contribution which appears as an additive term can be significant, especially when the amplitude of the wave is large.

3.2. *Dispersion relation in hot plasma (first-order temperature correction)*

Unless the plasma is extremely hot, we may use a perturbation technique to calculate the first-order temperature correction to the cold plasma result of the previous section. To do this, we first transform the Cartesian variables of integration in the expression of V_α to the frame S''_α which is moving with velocity $(0, 0, v_{\alpha 0})$ relative to S and then expand the integrand as a power series. The first-order correction is obtained by truncating the series at the quadratic terms.

The Lorentz transformations are

$$\xi''_\alpha = \xi_\alpha, \qquad \eta''_\alpha = \eta_\alpha, \qquad \zeta''_\alpha = \gamma_{\alpha 0}\left(\zeta_\alpha - \frac{v_{\alpha 0}}{c}\gamma_\alpha\right)$$

where

$$\gamma_{\alpha 0} = \left(1 - \frac{v_{\alpha 0}^2}{c^2}\right)^{-1/2} \quad \text{and} \quad \gamma''_\alpha = \gamma_{\alpha 0}\left(\gamma_\alpha - \frac{v_{\alpha 0}}{c}\zeta_\alpha\right).$$

Also $\mathrm{d}\zeta_\alpha = (\gamma_\alpha/\gamma''_\alpha)\,\mathrm{d}\zeta''_\alpha$. Therefore $\mathrm{d}\xi''_\alpha\,\mathrm{d}\eta''_\alpha\,\mathrm{d}\zeta''_\alpha/\gamma''_\alpha = \mathrm{d}\xi_\alpha\,\mathrm{d}\eta_\alpha\,\mathrm{d}\zeta_\alpha/\gamma_\alpha$. Further $N_{\alpha 0}F_{\alpha 0}(\xi''_\alpha, \eta''_\alpha, \zeta''_\alpha) = N_\alpha F_\alpha(\xi_\alpha, \eta_\alpha, \zeta_\alpha)$ where $N_{\alpha 0}F_{\alpha 0}$ is the equilibrium distribution function in S''_α. Also note that $N_\alpha = \gamma_{\alpha 0}N_{\alpha 0}$. The expression for V_α then becomes

$$V_\alpha = \iiint_{-\infty}^{\infty} \left\{1 + (\xi''_\alpha + \lambda_\alpha A_x)^2 + (\eta''_\alpha + \lambda_\alpha A_y)^2 + \left[\gamma_{\alpha 0}\left(\zeta''_\alpha + \frac{v_{\alpha 0}}{c}\gamma''_\alpha\right) + \lambda_\alpha A_z\right]^2\right\}^{1/2}$$
$$\times\left(1 + \frac{v_{\alpha 0}\zeta''_\alpha}{c\gamma''_\alpha}\right) F_{\alpha 0}(\xi''_\alpha, \eta''_\alpha, \zeta''_\alpha)\,\mathrm{d}\xi''_\alpha\,\mathrm{d}\eta''_\alpha\,\mathrm{d}\zeta''_\alpha. \tag{32}$$

Now expanding the coefficient of $F_{\alpha 0}$ in the integrand as a power series in ξ''_α, η''_α and ζ''_α and then performing integration term by term, assuming $F_{\alpha 0}$ to be isotropic, we obtain:

$$V_\alpha = \Delta_\alpha + \frac{\theta_\alpha}{2\Delta_\alpha}\left[1 + \gamma_{\alpha 0}\left(\gamma_{\alpha 0} + \frac{5v_{\alpha 0}}{c}\mathscr{L}_\alpha\right) + \frac{1 - u_{\alpha 0}^2\mathscr{L}_\alpha^2}{\Delta_\alpha^2}\right] \tag{33}$$

where

$$\theta_\alpha = \iint\int_{-\infty}^{\infty} (\xi_\alpha''^2, \eta_\alpha''^2, \zeta_\alpha''^2) F_{\alpha 0}\, \mathrm{d}\xi_\alpha''\, \mathrm{d}\eta_\alpha''\, \mathrm{d}\zeta_\alpha'' \equiv \frac{KT_\alpha}{m_\alpha c^2}. \tag{34}$$

Note that we have truncated the series at the quadratic terms, ignoring higher-order effects. Also Δ_α is the zero-order term, which is the result for the cold plasma. Further, on differentiating equation (33) we obtain

$$\frac{1}{\lambda_\alpha^2 \mathscr{A}} \frac{\partial V_\alpha}{\partial \mathscr{A}} = \frac{1}{\Delta_\alpha} - \frac{\theta_\alpha}{2\Delta_\alpha^3}\left[1 + \gamma_{\alpha 0}\left(\gamma_{\alpha 0} + \frac{5 v_{\alpha 0}}{c}\mathscr{L}_\alpha\right) + \frac{3(1 - u_{\alpha 0}^2 \mathscr{L}_\alpha^2)}{\Delta_\alpha^2}\right] \tag{35}$$

$$\frac{1}{\lambda_\alpha}\frac{\partial V_\alpha}{\partial A_z} = \frac{\mathscr{L}_\alpha}{\Delta_\alpha} - \frac{\theta_\alpha}{2\Delta_\alpha}\left(-5u_{\alpha 0} + \frac{(3\gamma_{\alpha 0}^2 + 5u_{\alpha 0}\mathscr{L}_\alpha - 1)\mathscr{L}_\alpha}{\Delta_\alpha^2} + \frac{3(1 - u_{\alpha 0}^2 \mathscr{L}_\alpha^2)\mathscr{L}_\alpha}{\Delta_\alpha^4}\right). \tag{36}$$

Since the analysis is correct only to the linear terms in θ_α, it is permissible to substitute for A_z in the coefficient of θ_α in equations (35) and (36). The substitution made is the expression given by cold plasma results, i.e. equations (28) and (30). With these approximations, the above equations become

$$\frac{1}{\lambda_\alpha^2 \mathscr{A}} \frac{\partial V_\alpha}{\partial \mathscr{A}} = \frac{1}{\Delta_\alpha} - P_\alpha \theta_\alpha \tag{37}$$

$$\frac{1}{\lambda_\alpha}\frac{\partial V_\alpha}{\partial A_z} = \frac{\mathscr{L}_\alpha}{\Delta_\alpha} - Q_\alpha \theta_\alpha \tag{38}$$

where

$$P_\alpha = \frac{1}{2(1 + \lambda_\alpha^2 \mathscr{A}^2)(1 + \Omega^2)^{3/2}} \times \left(5u_{\alpha 0}\Omega + \frac{(1 + \gamma_{\alpha 0}^2)(1 + \Omega^2) - 3u_{\alpha 0}^2 \Omega^2}{(1 + \Omega^2)(1 + \lambda_\alpha^2 \mathscr{A}^2)^{1/2}} + \frac{3}{(1 + \Omega^2)(1 + \lambda_\alpha^2 \mathscr{A}^2)^{3/2}}\right) \tag{39}$$

$$Q_\alpha = \frac{1}{2(1 + \lambda_\alpha^2 \mathscr{A}^2)^{1/2}(1 + \Omega^2)^{3/2}} \times \left(-5u_{\alpha 0} + \frac{[(3\gamma_{\alpha 0}^2 - 1)(1 + \Omega^2) - 3u_{\alpha 0}^2 \Omega^2]\Omega}{(1 + \Omega^2)(1 + \lambda_\alpha^2 \mathscr{A}^2)^{1/2}} + \frac{3\Omega}{(1 + \Omega^2)(1 + \lambda_\alpha^2 \mathscr{A}^2)^{3/2}}\right). \tag{40}$$

Now substituting the above equations in (15) and (16) we obtain

$$\frac{\mathscr{L}_\mathrm{e}}{\Delta_\mathrm{e}} = \frac{\mathscr{L}_\mathrm{i}}{\Delta_\mathrm{i}} + (Q_\mathrm{e}\theta_\mathrm{e} - Q_\mathrm{i}\theta_\mathrm{i}) \tag{41}$$

and

$$\frac{1}{\Delta_\mathrm{e}} + \frac{\mu}{\Delta_\mathrm{i}} - (P_\mathrm{e}\theta_\mathrm{e} + \mu P_\mathrm{i}\theta_\mathrm{i}) = \frac{\omega^2}{\omega_\mathrm{e}^2}. \tag{42}$$

The next step is to eliminate A_z so as to obtain ω in terms of the amplitude of the wave $\mathscr{A}$ only. In the circumstance that the waves are large amplitude, this is achieved by squaring (41), using (23) and continuing to work only to the linear terms in θ_α. After some algebra we obtain

$$\frac{1}{\Delta_e}+\frac{\mu}{\Delta_i}=-\frac{\mathscr{L}_i}{(1+\lambda_e^2\mathscr{A}^2)(1+\lambda_i^2\mathscr{A}^2)^{1/2}}(Q_e\theta_e-Q_i\theta_i)$$

where we have assumed $\mathscr{A}^2 \gg 1$, i.e. the waves are strong waves. Using the cold plasma expression for $\mathscr{L}_i$ we get

$$\frac{1}{\Delta_e}+\frac{\mu}{\Delta_i}=(1+\lambda_e^2\mathscr{A}^2)^{-1/2}\Omega(Q_i\theta_i-Q_e\theta_e) \tag{43}$$

so that

$$\frac{\omega^2}{\omega_e^2}=\left(\frac{\Omega Q_i}{(1+\lambda_e^2\mathscr{A}^2)^{1/2}}-\mu P_i\right)\theta_i-\left(\frac{\Omega Q_e}{(1+\lambda_e^2\mathscr{A}^2)^{1/2}}+P_e\right)\theta_e. \tag{44}$$

This gives the dispersion relation for the large amplitude waves. Notice that the zero-order (i.e. cold plasma) term does not survive for such large amplitude waves. This is also obvious from the right-hand side of equation (31). The dispersion relation in S′ is obtained by using (18):

$$n^2=1-\Gamma\frac{\omega_e'^2}{\omega'^2}\left[\left(\frac{\Omega Q_i}{(1+\lambda_e^2\mathscr{A}^2)^{1/2}}-\mu P_i\right)\theta_i-\left(\frac{\Omega Q_e}{(1+\lambda_e^2\mathscr{A}^2)^{1/2}}+P_e\right)\theta_e\right] \tag{45}$$

where $\Gamma=(1-n^2)^{-1/2}$. From the expressions for Q_α and P_α, it is evident that Q_i and P_i can be large compared to Q_e and P_e respectively. We may therefore conclude that the ionic contributions can be significant unless the ion temperature is negligibly small.

3.3. Dispersion relation for weak waves with first-order non-linear correction

In this section we treat the amplitude of the wave as a small parameter, and then use the perturbation technique to determine the dispersion relation incorporating a first-order non-linear correction, with the temperature in this case unrestricted. To be explicit, we shall expand V_α in powers of $\mathscr{A}$ and A_z and then truncate the series at terms of order $\mathscr{A}^3$. With this V_α, we calculate its differentials $\partial V_\alpha/\partial\mathscr{A}$ and $\partial V_\alpha/\partial A_z$ and then substitute them in equations (15) and (16). This will yield the desired dispersion relation.

For convenience we use the cylindrical polar coordinates $\boldsymbol{u}_\alpha=(\rho_\alpha\cos\phi_\alpha, \rho_\alpha\sin\phi_\alpha, \zeta_\alpha)$ and adopt the notation

$$\langle P(\rho_\alpha,\phi_\alpha,\zeta_\alpha)\rangle=\int P(\rho_\alpha,\phi_\alpha,\zeta_\alpha)F_\alpha(\boldsymbol{u}_\alpha)\,\mathrm{d}^3\boldsymbol{u}_\alpha$$

for an arbitrary function P of $\boldsymbol{u}_\alpha$.

Note that if P is independent of ϕ then

$$\langle P\cos^3\phi_\alpha\rangle=\langle P\cos\phi_\alpha\rangle=0, \qquad \langle P\cos^2\phi_\alpha\rangle=\tfrac{1}{2}\langle P\rangle$$

and

$$\langle P\cos^4\phi_\alpha\rangle=\tfrac{3}{8}\langle P\rangle.$$

We first observe that the potentials V_α (equation (5)) in the new notation take the form

$$V_\alpha=\langle D_\alpha\rangle \tag{46}$$

where

$$D_\alpha^2=\gamma_\alpha^2+\lambda_\alpha^2\mathscr{A}^2+\lambda_\alpha^2A_z^2+2\lambda_\alpha\mathscr{A}\rho_\alpha\cos\phi_\alpha+2\lambda_\alpha A_z\zeta_\alpha.$$

Therefore

$$\frac{1}{\lambda_\alpha}\frac{\partial V_\alpha}{\partial A_z}=\left\langle\frac{\lambda_\alpha A_z+\zeta_\alpha}{D_\alpha}\right\rangle \tag{47}$$

and

$$\frac{1}{\lambda_\alpha^2\mathscr{A}}\frac{\partial V_\alpha}{\partial\mathscr{A}}=\left\langle\frac{1+(\rho_\alpha\cos\phi_\alpha/\lambda_\alpha\mathscr{A})}{D_\alpha}\right\rangle. \tag{48}$$

Neglecting the terms of higher order than $\mathscr{A}^3$ it is found that

$$\begin{aligned}\frac{1}{D_\alpha}=\frac{1}{\gamma_\alpha}\bigg(1&-\frac{1}{\gamma_\alpha^2}\lambda_\alpha\mathscr{A}\rho_\alpha\cos\phi_\alpha-\frac{1}{2\gamma_\alpha^2}(\lambda_\alpha^2\mathscr{A}^2+2\lambda_\alpha A_z\zeta_\alpha)\\&+\frac{3}{2\gamma_\alpha^4}\lambda_\alpha^2\mathscr{A}^2\rho_\alpha^2\cos^2\phi_\alpha+\frac{3}{2\gamma_\alpha^4}\lambda_\alpha\mathscr{A}\rho_\alpha\cos\phi_\alpha(2\lambda_\alpha A_z\zeta_\alpha+\lambda_\alpha^2\mathscr{A}^2)\\&-\frac{5}{2\gamma_\alpha^6}\lambda_\alpha^3\mathscr{A}^3\rho_\alpha^3\cos^3\phi_\alpha\bigg)\end{aligned} \tag{49}$$

where we have assumed that A_z is of the order of $\mathscr{A}^2$. This assumption is indeed true for cold plasma, as may be seen from equation (28) and is verified *a posteriori* for the plasma (see equation (51)).

Hence, to the order $\mathscr{A}^2$

$$\frac{1}{\lambda_\alpha}\frac{\partial V_\alpha}{\partial A_z}=\left\langle\frac{\lambda_\alpha A_z}{\gamma_\alpha}+\frac{\zeta_\alpha}{\gamma_\alpha}\left(1-\frac{\lambda_\alpha^2\mathscr{A}^2+2\lambda_\alpha A_z\zeta_\alpha}{2\gamma_\alpha^2}+\frac{3\lambda_\alpha^2\mathscr{A}^2\rho_\alpha^2}{4\gamma_\alpha^2}\right)\right\rangle. \tag{50}$$

Here we observe that $\langle\zeta_i/\gamma_i\rangle=\langle\zeta_e/\gamma_e\rangle$ because from equation (1), it is clear that $\boldsymbol{J}_e+\boldsymbol{J}_i|_{\text{at }\boldsymbol{A}=0}=0$. That is:

$$\int\frac{\boldsymbol{u}_e}{\gamma_e}F_e(\boldsymbol{u}_e)\,d^3\boldsymbol{u}_e=\int\frac{\boldsymbol{u}_i}{\gamma_i}F_i(\boldsymbol{u}_i)\,d^3\boldsymbol{u}_i.$$

Therefore the relation (15) determines A_z as

$$\lambda_e A_z=\frac{\langle(\zeta_e/2\gamma_e^3)[1-(3\rho_e^2/2\gamma_e^2)]-(\mu^2\zeta_i/2\gamma_i^3)[1-(3\rho_i^2/2\gamma_i^2)]\rangle}{\langle[(1+\rho_e^2)/\gamma_e^3]+\mu[(1+\rho_i^2)/\gamma_i^3]\rangle}\lambda_e^2\mathscr{A}^2. \tag{51}$$

Also to the order $\mathscr{A}^2$, it is found that

$$\frac{1}{\lambda_\alpha^2\mathscr{A}}\frac{\partial V_\alpha}{\partial\mathscr{A}}=\left\langle\frac{1}{\gamma_\alpha}-\frac{\rho_\alpha^2}{2\gamma_\alpha^3}-\frac{\zeta_\alpha}{\gamma_\alpha^3}\left(1-\frac{3\rho_\alpha^2}{2\gamma_\alpha^2}\right)\lambda_\alpha A_z-\frac{1}{2\gamma_\alpha^3}\left(1-\frac{3\rho_\alpha^2}{\gamma_\alpha^2}+\frac{15}{8}\frac{\rho_\alpha^4}{\gamma_\alpha^4}\right)\lambda_\alpha^2\mathscr{A}^2\right\rangle. \tag{52}$$

Now using (51) and (52) in equation (16) we get

$$\omega^2/\omega_e^2=X_0-\tfrac{1}{2}X_1\lambda_e^2\mathscr{A}^2 \tag{53}$$

where

$$X_0=\left\langle\left(\frac{1}{\gamma_e}-\frac{\rho_e^2}{2\gamma_e^3}\right)+\mu\left(\frac{1}{\gamma_i}-\frac{\rho_i^2}{2\gamma_i^3}\right)\right\rangle \tag{54}$$

$$X_1 = \frac{\langle(\zeta_e/\gamma_e^3)[1-(3\rho_e^2/2\gamma_e^2)] - (\mu^2\zeta_i/\gamma_i^3)[1-(3\rho_i^2/2\gamma_i^2)]\rangle^2}{\langle[(1+\rho_e^2)/\gamma_e^3] + \mu[(1+\rho_i^2)/\gamma_i^3]\rangle}$$
$$+\left\langle\frac{1}{\gamma_e^3}\left(1-\frac{3\rho_e^2}{\gamma_e^2}+\frac{15}{8}\frac{\rho_e^4}{\gamma_e^4}\right)+\frac{\mu^3}{\gamma_i^3}\left(1-\frac{3\rho_i^2}{\gamma_i^2}+\frac{15}{8}\frac{\rho_i^4}{\gamma_i^4}\right)\right\rangle. \quad (55)$$

Note that the effects of the ionic motion stand out in the coefficient of μ and taking $\mu = 0$ gives the old results for the one-component plasma.

In the frame S′ the dispersion relation takes the form:

$$n^2 = 1 - \Gamma(X_0 - \tfrac{1}{2}X_1\lambda_e^2\mathscr{A}^2)(\omega_e'/\omega')^2. \quad (56)$$

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